Notes

Calculation of Temperatures in a Two-Layer Slab

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IN 1950, Jaeger[1] wrote " . . . the complete solutions of transient problems on composite walls, even in the simplest case of two layers only, are so very complicated as to be practically useless." When one uses a modern high-speed computing machine, this remark is no longer true. A Fortran program for the IBM 7090 was written recently for calculating temperatures in a two-layer slab by means of an analytical solution. It did not take long to write the program, and results are obtained fairly quickly. It is the purpose of the present paper to give some of the details of this study in order to encourage others to write similar programs for their computing machines.

The solution to the problem under consideration will be taken from Ref. 2 with certain modifications. For a two-layer slab it is convenient to have the origin of coordinates at the interface, so that a_1, b_1, a_2, and b_2 in Ref. 2 will be changed to $-s_1$, 0, 0, and s_2, respectively. Then x will be replaced by $-x$ in region 1. In addition, there will be changes in notation and trigonometric modification of equations. For example, κ_i, ξ_{in}, and γ_{in} there will be replaced by $1/\mu_i{}^2$, $\mu_i\omega_n$, and $\mu_i s_i\omega_n$ here. Finally, the problem will be generalized by including heat generation in each medium. With these revisions in mind, the problem may be formulated as follows:

$$\frac{\partial^2 U_i}{\partial x^2} = \mu_i{}^2 \frac{\partial U_i}{\partial t} - \frac{Q_i}{k_i} \quad 0 \le x < s_s \quad (1)$$

$$U_i(s_i,t) = \phi_i(t) \quad (2)$$

$$U_i(x,0) = F_i(x) \quad (3)$$

$$U_1(0,t) = U_2(0,t) \quad k_1\frac{\partial U_1(0,t)}{\partial x} = -k_2\frac{\partial U_2(0,t)}{\partial x} \quad (4)$$

where $i = 1$ for medium 1, and $i = 2$ for medium 2.

From Ref. 2, the solution to Eqs. (1–4) is found to be

$$U_i = \sum_{n=1}^{\infty} (A_{ijn}H_{in} - B_nH_{jn}) \sin\mu_i\omega_n(s_i - x) \quad (5)$$

where

$$2k_j = A_{ijn}[k_i\mu_i{}^2(k_1s_2 + k_2s_1) \cos^2\mu_i s_i\omega_n + k_j(k_1s_1\mu_1{}^2 + k_2s_2\mu_2{}^2) \sin^2\mu_i s_i\omega_n] \quad (6)$$

with $i = 1$ and $j = 2$, or else $i = 2$ and $j = 1$. Also,

$$2 = B_n[\mu_1\mu_2(k_1s_2 + k_2s_1) \cos\mu_1s_1\omega_n \cos\mu_2s_2\omega_n - (k_1s_1\mu_1{}^2 + k_2s_2\mu_2{}^2) \sin\mu_1s_1\omega_n \sin\mu_2s_2\omega_n] \quad (7)$$

$$H_{in}e^{\omega_n{}^2t} = \int_0^{s_i} \sin\mu_i\omega_n(s_i - \lambda) \times \left(\mu_i{}^2k_iF_i + \int_0^t e^{\omega_n{}^2\tau} Q_i \, d\tau\right) d\lambda + \mu_ik_i\omega_n \int_0^t \phi_i e^{\omega_n{}^2\tau} \, d\tau \quad (8)$$

The ω_n's are the positive roots of the equation

$$(k_1\mu_1 + k_2\mu_2) \sin\omega_n(\mu_1s_1 + \mu_2s_2) = (k_1\mu_1 - k_2\mu_2) \sin\omega_n(\mu_1s_1 - \mu_2s_2) \quad (9)$$

The first step in the calculations is to determine roots of Eq. (9). However, this is a fairly simple process because the sine curve on the right always has smaller amplitude and greater wavelength than the one on the left. There is exactly one root between every adjacent maximum and minimum of the sine curve on the left side of Eq. (9).

After an ω_n has been determined, the calculation of the corresponding term in the series for U_i is straightforward. The time required for computation is influenced by how many of the functions $F_1,F_2,Q_1,Q_2,\phi_1,\phi_2$ are not zero, and by whether the integrations in Eq. (8) can be performed analytically or must be done numerically. In one case, with $\phi_1 = 100 \sin t/4$ and the other five prescribed functions all zero, the time required to find 200 temperatures, using 50 terms of series (5) for each was 3.1 min. In this example, numerical integration of 10 steps was used in computing H by means of Eq. (8), even though the integration could have been performed analytically.

The program was checked in various ways. In one method, temperatures were calculated at several points in the composite slab at, for example, times 5 to 10 (in some units). The computed temperatures at $t = 5$ were then used as initial temperatures in a second example. Satisfactory agreements were obtained between results obtained in the two ways.

In another method of checking, temperatures were calculated as a function of the time at two or more places in the slab. Call two such places C and D. Then think of the slab as being mathematically sliced at one of the places, say at $x = D$. There is now a smaller slab having $x = D$ as an outer surface, and $x = C$ as an interior location. Temperatures are calculated at $x = C$ in this new, smaller slab, using the previously determined temperatures at $x = D$ as prescribed surface temperatures. Again the agreement was considered to be satisfactory.

It has not yet been assumed that the properties of the two media are related in any particular way. However, it will be found to be very desirable to deal with certain special cases whenever this can be done. For many purposes it should be possible to restrict one's calculations to such cases. In this way it will be found that one can reduce computing time, increase accuracy, and obtain a criterion for deciding whether enough terms have been taken in series (5).

Note first that, if $k_1\mu_1 = k_2\mu_2$, then the roots of Eq. (9) are precisely the roots of the left side of the equation. In this case, Lotkin[3] has proved that the composite slab problem can be reduced to that of a single slab. The greatest nontrivial simplification of the problem occurs in this case. However, simplifications also arise if $(\mu_1s_1 - \mu_2s_2)/(\mu_1s_1 + \mu_2s_2) \equiv \gamma$ is a rational number, since then the locations of the roots of Eq. (9) eventually repeat their pattern. To be useful, this repetition must take place within the number of terms that one is calculating. Thus, it is desirable to have γ equal to zero or to the ratio of two small integers.

If $\mu_1s_1 = \mu_2s_2$, the calculations become considerably easier, and the problem can be reduced to the solution of two problems in simple slabs by setting $V = U_1(x_1,t) - U_2(x_2,t)$ and $W = k_1\mu_1U_1(x_1,t) + k_2\mu_2U_2(x_2,t)$, where x_1 and x_2 are corresponding points in the two slabs such that $\mu_1x_1 = \mu_2x_2$.

As a case when the ratio of two small integers for γ would be a good approximation, let $0.12 \ \mu_1{}^2 = 1$, $0.41 \ \mu_2{}^2 = 1$, $s_1 = 2.3$, and $s_2 = 1.7$. Then, $\mu_1s_1/\mu_2s_2 \approx 2.5008$. The value 8.5 $(3/41)^{1/2}$ might just as well be selected for s_1. This is approximately 2.2993 and leads to $\mu_1s_1/\mu_2s_2 = \frac{5}{2}$ and to $\gamma = \frac{3}{7}$. This negligible change in s_1 produces a big simplification in the calculations. Equation (9) need now only be solved for the first three ω's, and the rest of the ω's can easily be obtained from them. Moreover, only the first three and the seventh A's and B's have to be computed. When determining the temperature at the interface, it is now found that every seventh term is zero and that, when the remaining terms are grouped three at a time, a convergent alternating series results. A criterion now exists for deciding whether

enough terms have been carried in the series, for if the individual terms are correct, the magnitude of the last group of three will be an error bound. A similar situation is found when calculating temperatures at other points.

References

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Flow in Pits of Fluid-Dynamic Origin

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A STUDY was made some years ago of the evidence of the nature of air flow over the surfaces of meteorites during their flight through air when they entered the earth's atmosphere. That evidence is left on the surface in the form of flow lines due to streaming of molten rock or iron oxide, just before the bodies in question were decelerated enough for ablation to cease.[1] In that study the suggestion was made, as based on the observations, that the shallow pits that have often been observed on meteorites, particularly iron ones, might have been caused by peculiarities of the air flow over the meteorite surface rather than by inhomogeneities of the meteorite material. In particular, it seemed likely that a pit would form in a surface if the flow that causes ablation can exist in the form of a "bound vortex," that is, a vortex with axis having a horseshoe shape which is held in one place on the ablating surface by the pressure gradients of the fluid flow. The pit would be formed at the apex of the horseshoe; the ends of the vortex would trail off to infinity in the wake. The horseshoe shape was concluded to be essential for the purpose of satisfying the continuity relation for the reversed flow implicit in any assumed vortex.

As a rather direct method for demonstrating that vortices of the type proposed actually exist over the surface of an ablating body and that, when they do, pits will be formed, tests were made by use of steady water flow over the surface of large salt blocks such as are fed to cattle. The relation between bound vortices and surface pits was verified in that pits were formed where bound vortices were predicted to exist, and not where they were absent. Figure 1 shows one of the pitted blocks. The evidence of the horseshoe shape of the bound vortex, presumed to have carved the pit, is clear in the shape of the lower one.

The question was posed how the ablation rate beneath a vortex could possibly be as great as that where no vortex exists; presumably, ablation rate depends on flow speed, which, in a vortex and especially inside a pit, certainly is less than it would be outside. An answer might be obtained if the flow about the pitted shape were studied in air flowing with the same Reynolds number as the water that shaped the block, under conditions of dynamic similarity.†

For measurement of the air flow inside the pit, a special velocity probe was made. It consisted of two lengths of 0.046-in. capillary tubing, each length closed at one end and pierced with a small hole in a side as near to the closed end as possible. The two tubes were soldered together with the
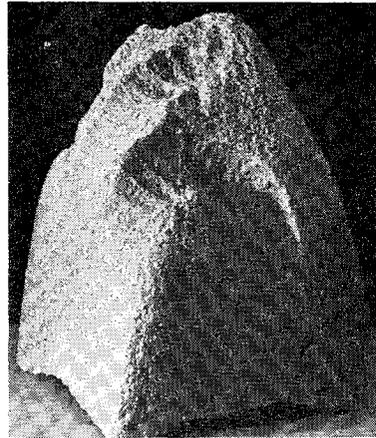
holes in the sides carefully set to face outward with the axes colinear. When the probe is set normal to a stream of air, the pressure difference between the two tubes can be used to measure the air speed; in order to determine its flow direction, similar measurements can be devised.

The probe was used to map, in three dimensions, the flow of air within the lower pit in the salt cake of Fig. 1, under conditions of dynamic similarity to the flow that carved the pit. The results are shown in Figs. 2 and 3, selected from more complete data in the forementioned dissertation. The curves show the flow at two stations in the deepest part of the pit. Station 3 is a little to the right of the pit vortex, and station 4 is somewhat further down toward the right.

Figures 2 and 3 show tangential and axial components of the air flow velocity at each station, as observed in a traverse carefully chosen to be normal to the surface of the pit. The traverse also passes through a point where the velocity component is zero in the plane normal to the pit axis. The pit axis is defined as a line within the pit, at every point of which the tangential velocity component vanishes; it is, in brief, the vortex axis.

It is noted that, in each of Figs. 2 and 3, the tangential velocity component varies nearly linearly with distance from the vortex axis. The axial component is small at station 3; it is large at station 4 and clearly downstream in direction. The data clearly demonstrate the existence of a horseshoe-shaped vortex within the pit, as predicted on the basis of the pit shape and flow markings.

Of special interest is the observation that the maximum tangential speed of the air flow inside the pit is substantially less than that outside the pit; its magnitude is about half that of the flow near the salt cake surface where no pit exists.
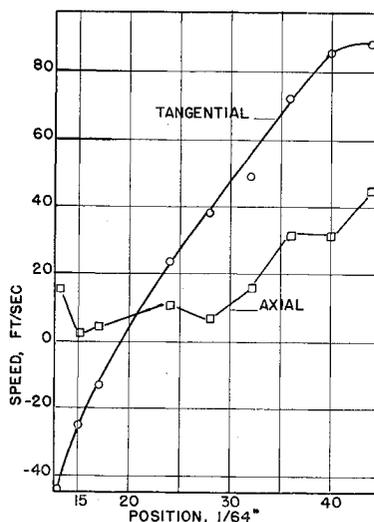

Fig. 2 Axial and tangential components of velocity of flow in the pit at station 3.